International Journal of Theoretical Physics, Vol. 30, No. 9, 1991

Invertible Point Transformations, Painlevé Test, and the Second Painlevé Transcendent

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Received October 18, 1990

The techniques of invertible point transformations and the Painlevé analysis can be used to construct integrable ordinary differential equations. We compare both techniques for the second Painlevé transcendent.

For nonlinear ordinary and partial differential equations the general solution usually cannot be given explicitly. It is desirable to have an approach to find out whether a given nonlinear differential equation is integrable. A powerful tool to find integrable differential equations (both ordinary or partial) is the Painlevé test (Weiss, 1984; Ward, 1984; Steeb and Euler, 1988; Steeb, 1990). A special class of ordinary differential equations which possess the so-called Painlevé property is given by the six Painlevé transcendents (Davis, 1962). They also arise in the group-theoretic reduction of soliton equations such as the Korteweg–de Vries equation, the modified Korteweg–de Vries equation, the one-dimensional sine-Gordon equation, and the one-dimensional nonlinear Schrödinger equation. For ordinary differential equations the Painlevé analysis and the invertible point transformation (Leach and Mahomed, 1985; Duarte *et al.*, 1987, 1989, 1990) can be used to construct integrable nonlinear equations or equations which are related to the Painlevé transcendents. We compare both techniques for the second Painlevé transcendents. A similar program for the anharmonic oscillator has been performed by Duarte *et al.* (1990).

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0020-7748/91/0900-1267$06.50/0

We consider the second Painlevé transcendent,

$$\frac{d^2U}{dT^2}=2U^3+TU+a \tag{1}$$

and perform an invertible point transformation to find the anharmonic oscillator

$$\frac{d^2u}{dt^2}+f_1(t)\frac{du}{dt}+f_2(t)u+f_3u^3=0 \tag{2}$$

This provides a condition on $f_1(t)$, $f_2(t)$, and $f_3(t)$ such that the anharmonic oscillator (2) can be transformed to the second Painlevé transcendent (1). We also perform the Painlevé test for equation (2). This also gives conditions on $f_1(t)$, $f_2(t)$, and $f_3(t)$. We show that the two conditions are the same for the two approaches.

Let us first discuss the Painlevé test. Euler *et al.* (1989) studied the anharmonic oscillator (2) where f_1, f_2, and f_3 are smooth functions of t with the help of the Painlevé test. We assume that $f_3 \neq 0$. For arbitrary functions f_1, f_2, and f_3 the nonlinear equation (2) cannot explicitly be solved. A remark is in order for applying the Painlevé test for nonautonomous systems. The coefficients that depend on the independent variable must themselves be expanded in terms of $t-t_1$, where t_1 is the pole position and we use the identity $t \equiv (t-t_1)+t_1$. If nonautonomous terms enter the equation at lower order than the dominant balance, the above-mentioned expansion turns out to be unnecessary, whereas if the nonautonomous terms are at dominant balance level, they must be expanded with respect to $t-t_1$. Obviously f_1, f_2, and f_3 enter the expansion not at dominant level.

Euler *et al.* (1989) gave the condition that (2) passes the Painlevé test. The condition is given by the differential equation

$$\begin{aligned}
&9f_3^{(4)}f_3^3-54f_3^{(3)}f_3'f_3^2+18f_3^{(3)}f_3^3f_1-36(f_3'')^2f_3^2+192f_3''(f_3')^2f_3\\
&\quad-78f_3''f_3'f_3^2f_1+36f_3''f_3^3f_2+3f_3''f_3^3f_1^2-112(f_3')^4+64(f_3')^3f_3f_1\\
&\quad+6(f_3')^2f_1'f_3^3-72(f_3')^2f_3^2f_2+90f_3'f_2'f_3^3-27f_3'f_1''f_3^3-57f_3'f_1'f_3^3f_1\\
&\quad+72f_3'f_3^3f_2f_1-14f_3'f_3^3f_1^3-54f_2''f_3^4-90f_2'f_3^4f_1+18f_1^{(3)}f_3^4\\
&\quad+54f_1''f_3^4f_1+36(f_1')^2f_3^4-36f_1'f_3^4f_2+60f_1'f_3^4f_1^2\\
&\quad-36f_3^4f_2f_1^2+8f_3^4f_1^4=0
\end{aligned} \tag{3}$$

where $f' \equiv df/dt$ and $f^{(4)} \equiv f'''' \equiv d^4f/dt^4$.

Now we ask whether the equation derived above can be found from (1) with the help of the invertible point transformation. Our invertible transformation is given by

$$T(u(t), t)) = G(u(t), t), \qquad U(T(u(t), t)) = F(u(t), t) \tag{4}$$

where

$$\Delta \equiv \frac{\partial G}{\partial t}\frac{\partial F}{\partial u} - \frac{\partial G}{\partial u}\frac{\partial F}{\partial t} \neq 0 \tag{5}$$

Since

$$\frac{dU}{dt} = \frac{dU}{dT}\frac{dT}{dt} = \frac{dU}{dT}\left(\frac{\partial T}{\partial u}\frac{du}{dt} + \frac{\partial T}{\partial t}\right) = \frac{\partial F}{\partial u}\frac{du}{dt} + \frac{\partial F}{\partial t} \tag{6}$$

and

$$\begin{aligned}\frac{d^2U}{dt^2} &= \frac{d^2U}{dT^2}\frac{dT}{dt}\left(\frac{\partial T}{\partial u}\frac{du}{dt} + \frac{\partial T}{\partial t}\right)\\ &\quad + \frac{dU}{dT}\left(\frac{\partial^2 T}{\partial u\,\partial t}\frac{du}{dt} + \frac{\partial^2 T}{\partial u^2}\left(\frac{du}{dt}\right)^2 + \frac{\partial T}{\partial u}\frac{d^2u}{dt^2} + \frac{\partial^2 T}{\partial t^2} + \frac{\partial^2 T}{\partial t\,\partial u}\frac{du}{dt}\right)\\ &= \frac{\partial^2 F}{\partial u\,\partial t}\frac{du}{dt} + \frac{\partial^2 F}{\partial u^2}\left(\frac{du}{dt}\right)^2 + \frac{\partial F}{\partial u}\frac{d^2u}{dt^2} + \frac{\partial^2 F}{\partial u\,\partial t}\frac{du}{dt} + \frac{\partial^2 F}{\partial t^2}\end{aligned} \tag{7}$$

we obtain from (1)

$$\frac{d^2u}{dt^2} + \Lambda_3\left(\frac{du}{dt}\right)^3 + \Lambda_2\left(\frac{du}{dt}\right)^2 + \Lambda_1\frac{du}{dt} + \Lambda_0 = 0 \tag{8}$$

where

$$\Lambda_3 = \left(-\frac{\partial F}{\partial u}\frac{\partial^2 G}{\partial u^2} + \frac{\partial^2 F}{\partial u^2}\frac{\partial G}{\partial u} - 2F^3\left(\frac{\partial G}{\partial u}\right)^3 - FG\left(\frac{\partial G}{\partial u}\right)^3 - a\left(\frac{\partial G}{\partial u}\right)^3\right)\Delta^{-1} \tag{9a}$$

$$\begin{aligned}\Lambda_2 = \Bigg(&-2\frac{\partial F}{\partial u}\frac{\partial^2 G}{\partial u\,\partial t} - \frac{\partial F}{\partial t}\frac{\partial^2 G}{\partial u^2} + \frac{\partial^2 F}{\partial u^2}\frac{\partial G}{\partial t} + 2\frac{\partial^2 F}{\partial x\,\partial t}\frac{\partial G}{\partial u} - 6F^3\left(\frac{\partial G}{\partial u}\right)^2\frac{\partial G}{\partial t}\\ &- 3FG\left(\frac{\partial G}{\partial u}\right)^2\frac{\partial G}{\partial t} - 3a\left(\frac{\partial G}{\partial u}\right)^2\frac{\partial G}{\partial t}\Bigg)\Delta^{-1}\end{aligned} \tag{9b}$$

$$\begin{aligned}\Lambda_1 = \Bigg(&-\frac{\partial F}{\partial u}\frac{\partial^2 G}{\partial t^2} - 2\frac{\partial F}{\partial t}\frac{\partial^2 G}{\partial u\,\partial t} + 2\frac{\partial^2 F}{\partial u\,\partial t}\frac{\partial G}{\partial t} + \frac{\partial^2 F}{\partial t^2}\frac{\partial G}{\partial u} - 6F^3\frac{\partial G}{\partial u}\left(\frac{\partial G}{\partial t}\right)^2\\ &- 3FG\frac{\partial G}{\partial u}\left(\frac{\partial G}{\partial t}\right)^2 - 3a\frac{\partial G}{\partial u}\left(\frac{\partial G}{\partial t}\right)^2\Bigg)\Delta^{-1}\end{aligned} \tag{9c}$$

$$\Lambda_0 = \left(-\frac{\partial F}{\partial t}\frac{\partial^2 G}{\partial t^2} + \frac{\partial^2 F}{\partial t^2}\frac{\partial G}{\partial t} - 2F^3\left(\frac{\partial G}{\partial t}\right)^3 - FG\left(\frac{\partial G}{\partial t}\right)^3 - a\left(\frac{\partial G}{\partial t}\right)^3 \right)\Delta^{-1} \quad (9d)$$

We now make a particular choice for F and G, namely

$$F(u(t), t) = f(t)u(t) \quad (10a)$$

$$G(u(t), t) = g(t) \quad (10b)$$

With this special ansatz we find that

$$\Lambda_3 = \Lambda_2 = 0 \quad (11)$$

and

$$\Lambda_1 = \frac{2f'g' - fg''}{g'f} \quad (12a)$$

$$\Lambda_0 = \frac{(f''g' - f'g'' - fg(g')^3)u - 2f^3(g')^3u^3 - a(g')^3}{g'f} \quad (12b)$$

For $a = 0$ it follows that

$$\frac{d^2u}{dt^2} + f_1(t)\frac{du}{dt} + f_2(t)u + f_3(t)u^3 = 0 \quad (13)$$

where

$$f_1 = \frac{2f'g' - fg''}{g'f} \quad (14a)$$

$$f_2 = \frac{g'f'' - f'g'' - fg(g')^3}{g'f} \quad (14b)$$

$$f_3 = -2(fg')^2 \quad (14c)$$

Here f and g are arbitrary functions of t.

For the case $a \neq 0$ one finds driven anharmonic oscillators. This case is not discussed here.

We are now able to eliminate f and g from system (14). We obtain

$$f(t) = Cf_3^{1/6}\exp\left\{\int \frac{f_1(t)\,dt}{3}\right\} \quad (15)$$

By inserting equations (15) and (14c) into equation (14b), we find

$$g=\frac{C^2 \exp(\frac{2}{3}\int f_1\,dt)}{18 f_3^{8/3}}(-6f_3 f_3'' + 7(f_3')^2 - 2f_1 f_3 f_3' - 12 f_1' f_3^2 + 36 f_2 f_3^2 - 8 f_1^2 f_3^2) \tag{16}$$

Inserting the derived f and g in equation (14a), we find condition (3).

To summarize: The condition (3) that the anharmonic oscillator (2) passes the Painlevé test is identical to the condition that the anharmonic oscillator (2) can be transformed to the second Painlevé transcendent.

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